Kybernetes
1983, Vol. 12 pp. 135–139

© Thales Publications (W.O.) Ltd.
Printed in Great Britain

PRESUPPOSITION IN BINARY AND FUZZY LOGICS*

RONALD R. YAGER

Machine Intelligence Institute, Iona College,
New Rochelle, New York 10801 (USA)

(*Received January 24, 1982*)

We introduce a four-valued logic that includes, in addition to true and false, the values unknown and non-existent. We introduce the idea of presupposition in fuzzy logic and then use this to relate this four valued logic to the binary logic.

INTRODUCTION

Logics are devised as a tool in which to help model and explain the ways people reason. In many situations the usual two-valued logic does not provide enough structure to capture necessary substitutes of human thought. In particular, it is sometimes necessary to introduce more valuations for truth associated with a proposition than the values *T* and *F*.

In devising a logical system there appears in many cases the need to include at least two other possible states of truth. These two states are the state of unknown and non-existence of any truth value. We shall first introduce a four-valued logic having these two additional states. We shall then discuss the fuzzy logic of Zadeh.[1] We shall then show the relationship between the fuzzy logic and this four-valued logic.

An important concept related the existence of a truth value is the concept of presupposition which will be discussed in this paper.

A FOUR-VALUED LOGIC

In this section we shall introduce a four-valued logic which we shall later relate to fuzzy logic.

Our first truth value shall be the value true, denoted *T*. The second truth value shall be the value false, denoted *F*. The third truth value shall be the value unknown, denoted *U*. The fourth truth value shall be the value non-existent, denoted *N*.

A careful distinction must be made between the values *U* and *N*. *U* or unknown implies the existence of some truth value *T* or *F* but its actual value is not known at this point. While *U* presupposes the existence of some truth value, *T* or *F*, the value non-existent, *N*, indicates the situation in which no truth value can be determined. It can find some use in situations in which there exists some internal inconsistencies in statement of a proposition. An example of this type of situation would be the statement "the woman elected president in 1980 is a democrat"[2] or "the life expediency of a rock is 10 years." This truth value *N* indicates that there is no existing truth value.

For this logic we provide the following basic definitions:

Negation

P	$-P$
T	F
U	U
F	T
N	N

If we denote "or" as ∨ then we get

$\vee$	T	U	F	N
T	T	T	T	N
U	T	U	U	N
F	T	U	F	N
N	N	N	N	N

---

* Tech. Report MII 1–09

If we define $\wedge$ = "and" then

$\wedge$	T	U	F	N
T	T	U	F	N
U	U	U	F	N
F	F	F	F	N
N	N	N	N	N

We can easily see that

$$P \wedge q = -[(-P) \vee (-q)]$$

If we define implication as

$$P \rightarrow q \underset{\text{df}}{=} -P \vee q$$

then its truth value is

$P \rightarrow q$	T	U	F	N
T	T	U	F	N
U	T	U	U	N
F	T	T	T	N
N	N	N	N	N

We can see that our logic satisfies the following rules:

(1) Communativity of $\vee$, $\wedge$

$$P \vee q = q \vee P$$
$$P \wedge q = q \wedge P$$

(2) Associativity of $\vee$, $\wedge$

$$(P \vee q) \vee r = P \vee (q \vee r)$$
$$(P \wedge q) \wedge r = P \wedge (q \wedge r)$$

(3) De Morgans Laws

$$(P \wedge q) = -[(-P) \vee (-q)]$$
$$(P \vee q) = -[(-P) \wedge (-q)]$$

(4) Involution

$$-(-P) = P$$

(5) If P and $P \rightarrow q$ are both true than q is true.

(6) If P is true then $P \vee q$ is true if a truth value for q exists (q doesn't have truth value N).

FUZZY LOGIC

The continuous multivalued logic assumes a set $I = [0, 1]$ of possible truth values which can be associated with a proposition. The larger the value in I the more truthful.

If P is a proposition such that the truth of P, denoted $T(P)$, is $T(P) = a$, then T (not P) $= T(\sim P) = 1 - a$. If P and Q are two propositions having truth values $T(P) = a$ and $T(Q) = b$ then

$$T(P \text{ and } Q) = \min[T(P), T(Q)] = a \wedge b$$

and

$$T(P \text{ or } Q) = \max[T(P), T(Q) = a \vee b$$

The implication operation can be defined in this logic by

$$T(P \rightarrow Q) = T(\text{not } P \text{ or } Q) = (1 - a) \vee b$$

or

$$T(P \rightarrow Q) = \min(1, 1 - a + b).$$

The introduction of this multivalued logic provides an extension of the usual two valued logic [0, 1] by allowing degrees of truth between 1 and 0, true and false. However, in using this logical structure every proposition must be given a value in the set I. Thus this structure presupposes the existence of some truth value for every proposition as does the ordinary two-valued logic although in this case we have a richer class to choose from.

Zadeh[1] has introduced the idea of fuzzy truth values and the associated idea of a fuzzy logic based upon these fuzzy truth values. Yager[3] has studied these ideas in great detail.

The idea of a fuzzy truth value is a generalization of the concept of a truth value selected from the multivalued set I. Essentially a fuzzy truth values allows us to associate linguistic concepts to truth values. The introduction of fuzzy subsets to represent truth values allows for the introduction of some uncertainty in the selection of the value of I to represent the truth associated with a proposition P.

If P is a proposition having as its truth value the fuzzy truth H, a fuzzy subset of I that is $T(P) = H$, where

$$H: \rightarrow I \rightarrow I$$

such that for each $a \in I$, $H(a)$ indicates the possibility that the truth of P is a. Thus a fuzzy truth value associates a possibility distribution over the set I of available truth values.

The truth values of the multivalued logic I can be represented as special cases of the class of all fuzzy truth values of I. For example, the truth value $T(P) = a$, where $a \in I$, can be represented as the fuzzy subset

$$a = \left\{\frac{1}{a}\right\}$$

In this case the possibility that the truth of the proposition is a 1 and the possibility that it is anything else is zero. Thus the elements of the multivalued set I are representable as fuzzy truth values in which there is absolute certainty as to the value associated with $T(P)$.

Another special case is the situation in which $T(P)$ is any value in I, in this case $T(P) = I$. Furthermore, the case where $T(P)$ can be any value less that 0.3, can be represented as $T(P) = [0, 0.3]$.

However, in addition to allowing us the ability to introduce uncertainty into the assignation of truth values for a proposition, the introduction of fuzzy truth values provides a mechanism for introducing the idea of presupposition or existence of any truth value at all. This mechanism is not available in the ordinary multivalue logic based upon I.

Def: Assume P is a proposition which has a fuzzy truth value A, $T(P) = A$, the degree of presupposition or existence of some value of I as the associated truth value of P is defined as

$$S(A) = \max_{x \in I} A(x).$$

Thus $S(A)$ measures the maximum possibility of existence of any truth value associated with P.

For any normal fuzzy truth value, a fuzzy set with at least one element with membership grade 1, $S(A) = 1$. Furthermore, we note that a subnormal fuzzy truth value has $S(A) < 1$.

A distinction must be made between the fuzzy truth value $A = 0 = \{0\} = \{1/0\}$ and the value $A = \emptyset$. The first case is that of a truth value which absolutely false and $S(A) = 1$. In the second case $S(A) = 0$ and we have the situation were we have a zero presupposition of any value of I being the truth of P.

This idea of associating the maximum grade of membership of a fuzzy truth value is a generalization of the concept of presupposition presented by Bergman.[2]

We can easily see that

$$S(A) = 0 \text{ iff } S = \emptyset,$$

thus our presupposition is zero iff A is the null set.

By use of the extension principle we can define the logical operations, "not", "and," "or", and "implication" for propositions having fuzzy truth values.

In the following we shall assume that p and q are two propositions having the fuzzy truth values A and B respectively, $T(p) = A$ and $T(q) = B$. The truth value of $T(\text{not } p)$ is defined as the fuzzy truth value C such that for any $x \in I$

$$C(x) = A(1 - x).$$

Example: If

$$A = \left\{\frac{1}{1}, \frac{0.6}{8}, \frac{0.4}{3}, \frac{0}{0}\right\}$$

then

$$C = \left\{\frac{1}{0}, \frac{0.6}{2}, \frac{0.4}{7}, \frac{0}{1}\right\}$$

Note:

If $A = \left\{\frac{1}{a}\right\} = a$ then $C = \left\{\frac{1}{1 - a}\right\} = 1 - a$;

thus for values from the multivalued logic the fuzzy based operations give the same result as if we used the usual multivalue logic rules.

Note:

Since $\max_{x \in I} A(x) = \max_{x \in I} A(1 - x)$, the degree of presupposition assigned a truth value associated with a proposition not p is the same as associated with the proposition P.

The truth of the proposition "p and q", denoted $T(p \text{ and } q)$ is defined to be the fuzzy truth value D, where

$$D = \bigcup_{x, y \in I} \left\{\frac{A(x) \wedge B(y)}{x \wedge y}\right\}$$

Example:

$$A = \left\{\frac{1}{1}, \frac{0.2}{0.3}\right\}$$

$$B = \left\{\frac{1}{0.7}, \frac{0.2}{0.5}\right\}$$

$$T(p \text{ and } q) = \left\{\frac{1}{0.7}, \frac{0.2}{0.5}, \frac{0.2}{0.3}\right\}$$

Note: The degree of presupposition associated with the truth value D, $S(D)$ can easily be seen to be

$$S(D) = \left[\max_{x \in I} A(x) \wedge \max_{y \in I} B(y)\right]$$

From this it follows that $S(D) = 1$ iff both $S(A)$ and $S(B) = 1$, that is A and B are both normal fuzzy truths.

If either A or B are the null set $S(D) = 0$. Furthermore, from our definition of D we note that $D(y) = 1$ iff there exist two values x, $y \in I$ such that $A(x) = B(y) = 1$ and $\min(x, y) = 3$. Thus $D(1) = 1$ iff $A(1) = B(1) = 1$ and $D(0) = 1$ if one set is normal, has presupposition 1, and the other set has membership grade 1 for 0.

The truth of the proposition, "P or q", denoted $T(p \text{ or } q)$ is defined to be the fuzzy truth value G, where

$$G = \bigcup_{x, y \in I} \left\{\frac{A(x) \wedge B(y)}{x \vee y}\right\}$$

Example:

$$A = \left\{\frac{1}{1}, \frac{0.6}{0.3}\right\}$$

$$B = \left\{\frac{1}{0.7}, \frac{0.2}{0.5}\right\}$$

$$G = \left\{\frac{1}{1}, \frac{0.6}{0.7}, \frac{0.2}{0.5}\right\}$$

We see easily again for this case

$$S(G) = \left[\max_{x \in I} A(x) \wedge \max_{y \in I} B(y)\right] = S(A) \wedge S(B).$$

Thus $S(G) = 1$ iff both A and B have presuppositions of 1.

The truth of the proposition $p \longrightarrow q$ denoted $T(p \longrightarrow q)$ is defined to be the fuzzy set H where

$$H = \bigcup_{x, y \in I} \left\{\frac{A(x) \wedge B(y)}{x \longrightarrow y}\right\}$$

and where $x \longrightarrow y$ is an implication in the multivalued logic.

Note: Regardless of how we define $x \longrightarrow y$

$$S(H) = \left[\max_{x \in I} A(x)\right] \wedge \left[\max_{y \in I} B(y)\right] = S(A) \wedge S(B)$$

Th Assume $f(A_1, A_2, \ldots A_n)$ is some logical sentence involving the usual connectives "not", "and", "or" and "$\longrightarrow$" and $A_1, \ldots A_n$ are fuzzy truth values than

$$S(f) = \min_i S(A_i).$$

Thus any sentence involving fuzzy truth values has presupposition equal to the smallest of any of the elements which comprise the sentence.

PRESUPPOSITION IN BINARY LOGIC

In this section we should like to try to introduce the idea of presupposition in the binary logic and show its relation to our hypothesized four valued logic.

If we consider the truth space

$$X = \{0, 1\}$$

then this set generates four subsets:

$$A = \{1\}$$
$$B = \{0, 1\}$$
$$C = \{0\}$$
$$D = \Phi.$$

It is our contention that these four subsets correspond to the truth values in our original four valued logic as follows:

$$T = A$$
$$U = B$$
$$F = C$$
$$N = D.$$

In order to show this correspondence we must show that the truth table involving A, B, C, D correspond to those given for this logic.

First we note that each of the above sets can be represented as fuzzy sets.

$$A = \left\{\frac{0}{0}, \frac{1}{1}\right\}$$

$$B = \left\{\frac{1}{0}, \frac{1}{1}\right\}$$

$$C = \left\{\frac{1}{0}, \frac{0}{1}\right\}$$

$$D = \left\{\frac{0}{0}, \frac{0}{1}\right\}$$

We note that $S(A) = S(B) = S(C) = 1$ and $S(0) = 0$, so that D corresponds to a situation where we have non-existence of any truth value in X. Using the rules of fuzzy logic we see that

$$\text{not } A = \left\{\frac{0}{1}, \frac{1}{0}\right\} = C$$

$$\text{not } B = \left\{\frac{1}{1}, \frac{1}{0}\right\} = B$$

$$\text{not } C = \left\{\frac{1}{1}, \frac{0}{0}\right\} = A$$

$$\text{not } D = \left\{\frac{0}{1}, \frac{0}{0}\right\} = D$$

This relationship corresponds exactly to that for our original logic. We can easily see that

$$\left.\begin{matrix} D \vee A \\ D \vee B \\ D \vee C \\ D \vee D \end{matrix}\right\} = \left\{\frac{0}{1}, \frac{0}{0}\right\} = D$$

$$\left.\begin{matrix} A \vee A \\ A \vee B \\ A \vee C \end{matrix}\right\} = \left\{\frac{0}{0}, \frac{1}{1}\right\} = A$$

$$\left.\begin{matrix} B \vee B \\ B \vee C \end{matrix}\right\} = \left\{\frac{1}{0}, \frac{1}{1}\right\} = B$$

$$C \vee C = \left\{\frac{1}{0}, \frac{0}{1}\right\} = C.$$

Thus A, B, C, D do indeed act like T, U, F, N respectively under "or." In similar manner we can show that A, B, C, D act like T, U, F, N respectively under both "and" and "implication."

REFERENCES

1. L. A. Zadeh, "Fuzzy logic and approximate reasoning" *Synthese* 30, 407–428 (1975).
2. M. Bergman, "Ony, even and clefts in two-dimensional logic" *Proc. IEEE Symp. Multiple Valued Logic*, Okla. City, **117**, 23 (1981).
3. R. R. Yager, "Strong truth and rules of inference in fuzzy logic and approximate reasoning" *Systems and Cybernetics* (to appear).